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STOCHASTIC APPROACHES FOR DAMAGE EVOLUTION IN STANDARD AND NON-STANDARD CONTINUA

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Abstract—Damage evolution in quasi-brittle materials is a complex process in which heterogeneity plays an important role. This heterogeneity may imply that the exact failure mode can be highly dependent upon the precise spatial distribution of initial imperfections. To model this inhomogeneity stochastic distributions of material properties must be used in numerical simulations. However, the use of a stochastic approach does not resolve the issue of the change of character of the governing differential equations during progressive damage. To avoid such a change of character higher order terms, either in space or in time, must be added to the standard continuum description (regularization techniques). A simulation technique that describes the failure process properly must incorporate both a regularization technique and a stochastic description of the disordered continuum. This statement will be substantiated here by presenting finite element analyses of direct tension tests with a standard local damage model and with a nonlocal damage model. The randomness in the damage process will be introduced by considering the initial damage threshold of the continuum damage model as a random field, characterized by a relevant distribution and autocorrelation coefficient function. The response statistics calculated by the Monte-Carlo technique will be presented for two different levels of finite element discretization. The nonlocal and random field formulations both rely on the introduction of a length parameter: the internal length scale in case of the nonlocal continuum and the correlation length for the random field. The effect of the relative variation of the correlation length and the internal length scale will also be discussed.

1. INTRODUCTION

Failure in quasi-brittle disordered materials involves a progressive concentration of deformation in narrow zones. This phenomenon is referred to as strain localization. Strain localization is initially driven by the growth of existing micro-defects in the disordered continuum and the development of several non-critical damage zones. In a second stage, one critical damage zone usually develops, such as cracks in concrete, shear bands in soils and rock faults.

In a deterministic continuum approach the microstructural changes during the damage process can be modeled by means of a strain-softening material behavior, that is, after a certain limit, for example, the tensile strength f_t, or equivalently the initial damage threshold level $K_0 = f_t/E$, a descending relation between stress and strain is adopted (Fig. 1). An inherent property of a constitutive law with a softening branch, when applied in a standard, rate-independent continuum, is that the damage tends to localize in a discrete line or plane. Intimately related is the fact that the field equations cease to be elliptic at the onset of localization and the mathematical model can no longer be a proper description of progressive damage processes (de Borst *et al.*, 1993). The ensuing loss of well-posedness of the rate boundary value problem causes a severe mesh sensitivity in finite element analyses. To overcome this deficiency of the standard continuum, two different approaches can be used. Firstly, we have discontinuum approaches where the crack is modeled by interface elements

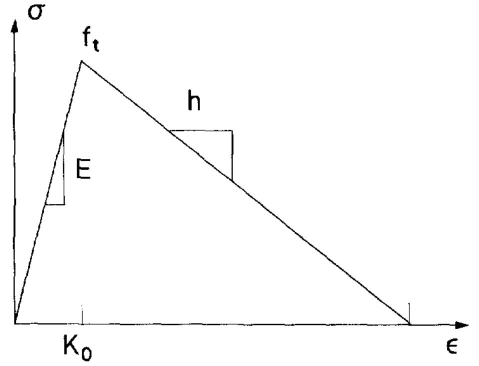


Fig. 1. Stress–strain diagram for linear strain-softening behavior.

showing a softening force-displacement behavior (Rots, 1988). Alternatively, the standard continuum model can be enriched. These non-standard models, also referred to as regularization methods, include the nonlocal damage model (Pijaudier-Cabot and Bažant, 1987), the Cosserat continuum (de Borst, 1991) and rate dependent and gradient models (de Borst *et al.*, 1993).

An important property of quasi-brittle materials is the inherent heterogeneity and the presence of initial damage. When these uncertainties have a limited influence on the global response of the structure, deterministic non-standard continuum models are generally sufficient. These non-standard continua incorporate an internal length scale l which measures the distance over which a strong micro-structural interaction will exist. Although the internal length scale l is thought to reflect the physical phenomenon of randomly distributed interacting damage processes at a microlevel, the parameter is generally considered as a deterministic property. By the introduction of an internal length scale l the non-standard continuum models are capable of describing the deterministic size effects observed in many experiments.

When the damage evolution, the structural behavior and reliability of mechanical structures are highly directed by the internal disorder of the material, it is no longer possible to avoid the modeling of the micro-structural uncertainties and their interactions. A possible approach is based upon the mapping of the micro-structural geometry and topology onto a discrete representation and then the use of the finite element method to perform numerical simulations (Breysse *et al.*, 1993). However, these models suffer from objectivity regarding the finite element discretization and will exhibit spurious results.

When the aim is to analyze the damage behavior of real engineering structures, a more appropriate approach to account for material disorder is to consider the macro-scale material properties to be randomly distributed in space (Carmeliet and Hens, 1994). A suitable framework to describe the continuous spatial distribution of the random variables and their interdependence is the random field theory (e.g. Vanmarcke, 1983). This approach introduces a second length parameter, the correlation length θ, which describes the correlative characteristics of the random continuum.

Nevertheless, the use of the random field theory does not resolve the above-mentioned issue of loss of well-posedness of the rate boundary value problem during progressive damage, as occurs in standard continua. It will be shown that a numerical technique that describes the failure process properly within the framework of continuum mechanics must incorporate both a regularization of the standard continuum and a stochastic description of the disordered continuum.

The paper starts by reviewing the governing equations of the deterministic nonlocal continuum damage model. It will be shown that the nonlocal theory provides a proper regularization technique to capture strain localization. In the next section the stochastic approach based on the random field theory is laid out. The initial damage in the continuum is modeled as a univariate homogeneous isotropic autocorrelated two-dimensional random field. Finally the role of the length parameters, the internal length l and the correlation length θ, is discussed.

2. DETERMINISTIC APPROACH

A possible way to remedy the ill-posedness of the rate boundary value problem at the onset of localization is the nonlocal damage concept. In the nonlocal theory the constitutive equations are formulated such that they allow for dependence on the variables of the whole body (Edelen, 1976). In the nonlocal damage model as developed by Pijaudier-Cabot and Bažant (1987), the nonlocal concept is only applied to the internal damage variable D, while the other variables remain local. In this section, we will show that application of the nonlocal concept to damage can regularize the rate boundary value problem.

The constitutive law of the continuum with isotropic damage may be written in the form:

$$\boldsymbol{\sigma} = (1-D)\,\mathbf{C}\boldsymbol{\varepsilon} \tag{1}$$

where $\mathbf{C}$ is the initial elasticity tensor of the virgin material, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ the stress and strain tensors and D the internal damage variable. From an initial equilibrium state given by the stress $\boldsymbol{\sigma}_0$, the corresponding strain $\boldsymbol{\varepsilon}_0$ and value of damage D_0, the rate constitutive relation can be derived as:

$$\dot{\boldsymbol{\sigma}} = (1-D_0)\,\mathbf{C}\dot{\boldsymbol{\varepsilon}} - \dot{D}\mathbf{C}\boldsymbol{\varepsilon}_0 \tag{2}$$

with the conditions for damage growth $\dot{D}$:

$$\begin{aligned} &\text{if } f(\bar{\varepsilon}) = 0 \text{ and } \dot{f}(\bar{\varepsilon}) = 0 \quad \text{then } \dot{D}(x) = \frac{\partial F(\bar{\varepsilon})}{\partial \bar{\varepsilon}}\dot{\bar{\varepsilon}} \\ &\text{if } f(\bar{\varepsilon}) < 0 \text{ or if } f(\bar{\varepsilon}) = 0 \text{ and } \dot{f}(\bar{\varepsilon}) < 0 \quad \text{then } \dot{D}(\mathbf{x}) = 0 \end{aligned} \tag{3}$$

with F the evolution law for damage and f the damage loading function:

$$f(\bar{\varepsilon}) = \bar{\varepsilon} - K(\bar{\varepsilon}). \tag{4}$$

K is the damage threshold. In the initial state the damage threshold K is taken equal to the initial damage threshold K_0. Afterwards K is assumed to be the maximum value of $\bar{\varepsilon}$ ever reached at the considered point $\mathbf{x} = [x, y, z]^{\mathrm{T}}$ of the solid during the loading history. The damage loading function f and damage evolution F are specified through the nonlocal strain measure $\bar{\varepsilon}$. The definition of $\bar{\varepsilon}$ is based on the hypothesis of the attenuating neighborhood. It is known that in quasi-brittle materials the effect of long-range interactions on the local damage evolution will attenuate fast with the distance. Accordingly:

$$\bar{\varepsilon}(\mathbf{x}) = \frac{1}{V_r}\int_v \tilde{\varepsilon}(\mathbf{x}+\boldsymbol{\tau})\alpha(\boldsymbol{\tau})\,\mathrm{d}V \tag{5}$$

with $\boldsymbol{\tau} = [\tau_x, \tau_y, \tau_z]^{\mathrm{T}}$ the separation vector between two points $\mathbf{x}$ and $\mathbf{x}+\boldsymbol{\tau}$, and α an attenuating weight function. We note that the weight function must be subjected to continuity (smoothness) conditions, in order to avoid unacceptable fluctuations of the damage variable. Motivated by this, a squared exponential weight function is introduced, which remains continuous in all points:

$$\alpha(\boldsymbol{\tau}) = \mathrm{e}^{-[|\boldsymbol{\tau}|^2/2l^2]} \tag{6}$$

with l the so-called internal length scale. In Pijaudier-Cabot and Benallal (1993), it is shown that the squared exponential weight function (in contrast to a uniform weight function) admits a non-trivial solution to the bifurcation problem at the onset of localization. In eqn (5), V_r is the representative volume which for the infinite domain reads:

$$V_r = \int_V \alpha(\tau)\,\mathrm{d}v. \tag{7}$$

The local equivalent strain $\tilde{\varepsilon}$ in eqn (5) is defined as the accumulated tensile strain in the material (Mazars, 1984; Mazars and Pijaudier-Cabot, 1989):

$$\tilde{\varepsilon} = \sqrt{\sum_{i=1}^{3} (\langle \varepsilon_i \rangle_+)^2},$$
$$\langle \varepsilon_i \rangle_+ = \varepsilon_i \quad \text{if } \varepsilon_i > 0,$$
$$\langle \varepsilon_i \rangle_+ = 0 \quad \text{if } \varepsilon_i \leqslant 0, \tag{8}$$

where ε_i are the principal strains.

To demonstrate the essential features of the nonlocal damage model, we solve the two-dimensional plane stress problem of a direct tensile test. A linear-elastic linear-softening behavior has been assumed: Young's modulus $E = 20{,}000$ MPa, tensile strength $f_t = 2$ MPa, initial damage threshold $K_0 = 1 \times 10^{-4}$, softening modulus $h = -0.1E$ (Fig. 1). The specimen has a length of 100 mm and a width of 25 mm. To initiate and to promote localization of damage in the middle of the specimen an imperfect zone is assumed. For the present case a 1% reduction of the value of the initial damage threshold K_0 over a constant width of 6.25 mm is sufficient. A value of the internal length scale l of 5 mm has been assumed. The effect of the mesh refinement on the response is studied using three meshes: 32×8, 64×16 and 128×32 elements. Figure 2, which gives the overall stress–displacement curves, shows that the nonlocal results are completely insensitive with respect to the discretization. The simulations with 256, 1024 and 4096 elements yield exactly the same result. These results point out that the nonlocal concept regularizes the standard local continuum and properly captures the phenomenon of damage localization. For comparison the stress–displacement curve for an internal length scale of 2.5 mm has been calculated. A more brittle response is observed. The reason for this becomes clear when we compare the evolution of the damage distribution along the axis of the tensile specimen for internal length scales of 5 and 2.5 mm, respectively (Fig. 3). The internal length scale l clearly governs the width of the damage zone. The same conclusions can be drawn for other non-standard continuum models such as the gradient model (e.g. Pamin, 1993). All these models share the same essential property: the introduction of an internal length scale, which forces the energy dissipation to remain finite but nonzero and avoids spurious and discontinuous localization.

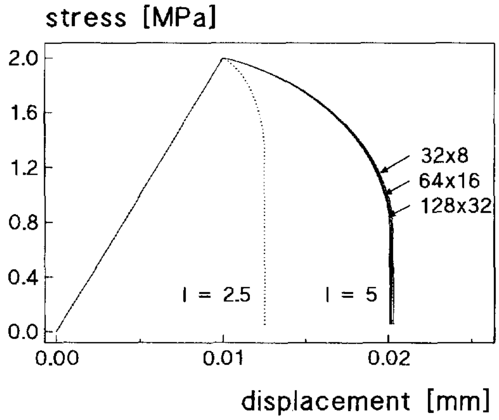


Fig. 2. Mesh sensitivity of stress–displacement diagram for tensile test. Influence of the internal length scale l.

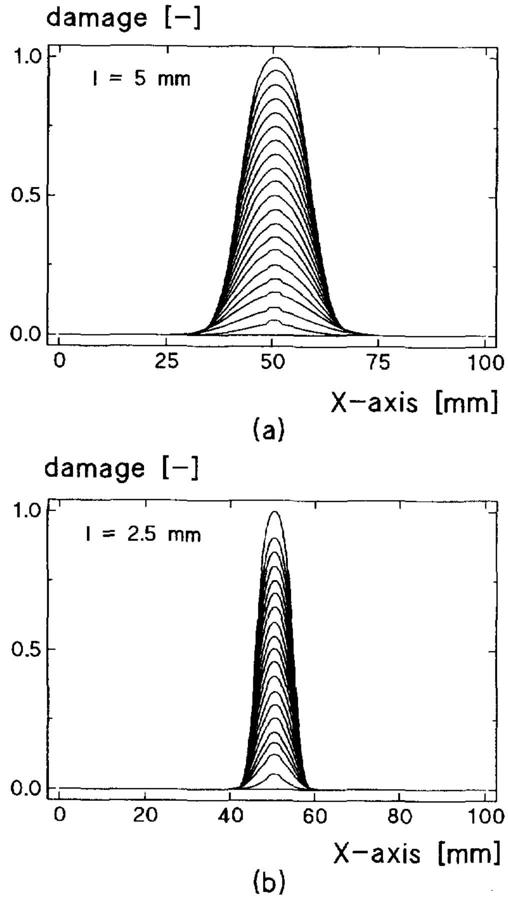


Fig. 3. Damage profiles along the longitudinal symmetry axis of the tensile specimen : (a) $l = 5$ mm ; (b) $l = 2.5$ mm.

3. STOCHASTIC APPROACH

As mentioned in the Introduction, heterogeneity may play an important role in the localization process in quasi-brittle materials. Therefore, the use of stochastic approaches seems essential. They give useful information in the form of higher order moments or full distributions of the intrinsic properties regarding failure: failure strength and dissipation of energy (e.g. Carmeliet and Hens, 1992). They also offer a framework for analyzing failure probability or reliability of mechanical structures. Another use is the study of statistical size, volume and shape effects not captured by the deterministic gradient or nonlocal approach (Mazars *et al.*, 1991). A fundamental question regarding application to localization phenomena is: "*Does a statistical description of the standard continuum resolve the ill-posedness of the continuum model after the onset of localization*?" This question becomes imperative, especially if we consider that the description of a heterogeneous continuum by correlated random variables introduces a length parameter in the form of the correlation length θ analogous to the introduction of an internal length scale in non-standard continuum approaches. The correlation length θ is a measure for the rate of fluctuations of the random field and may significantly influence the damage process and global response of the structure. The example of a tensile specimen with random initial damage is well suited for studying this fundamental issue. We assume that the initial damage threshold is randomly distributed over the solid and can be represented by a non-Gaussian correlated random field. For the non-Gaussian field a three-parameter extreme value distribution type III function is assumed:

$$f_{K_0}(K_0) = \lambda\mu(K_0 - K_0^{\min})^{\mu-1} \exp[-\lambda(K_0 - K_0^{\min})^{\mu}] \tag{9}$$

with $K_0^{\min}$ the lower bound of the initial damage threshold. The choice of this distribution is based on the following physical considerations: (i) the initial damage threshold K_0 will show a finite lower bound rather than an infinite lower tail; (ii) the initial damage will very likely be asymmetrically distributed, and (iii) the initial damage in a macroscopic sense is strongly influenced by the largest microstructural flaw and will therefore follow an extreme value distribution. The material data are taken from Carmeliet (1992) and have been assigned the following values: $\lambda = 6.56 \times 10^5$, $\mu = 1.6$, $K_0^{\min} = 0.66 \times 10^{-4}$. An inverse fitting procedure, comparing the numerical responses to the experimental distribution of tensile strength and energy dissipation, was used for the proper identification of the material parameters (for more information, see Carmeliet, 1992; Carmeliet and Hens, 1992). We assume the field to be homogeneous and isotropic, which implies that the autocorrelation coefficient function $\rho(\boldsymbol{\tau})$ can be expressed in terms of the separation vector $\boldsymbol{\tau} = [\tau_x, \tau_y, \tau_z]^T$ between the points $\mathbf{x}$ and $\mathbf{x}+\boldsymbol{\tau}$. The autocorrelation coefficient function is assumed to be of the same form as the weight function of the nonlocal damage model:

$$\rho(\boldsymbol{\tau}) = e^{-(|\boldsymbol{\tau}|^2/2d^2)} \tag{10}$$

with d a parameter, which in the case of a squared exponential function as in eqn (10) is related to the correlation length θ by $d = \theta/\sqrt{2}$. The correlation length θ is defined here as the length over which the autocorrelation coefficient function drops to a small value, say e^{-1}.

For finite element analysis involving random field properties, it is necessary to discretize the continuous random field into random vector representations. This discretization involves the division of the structure into several stochastic elements and the representation of the stochastic field within the elements by random variables. A critical review of existing discretization methods has been presented by Li and Kiureghian (1993). In this paper, we will use the midpoint method (Der Kiureghian and Ke, 1988). To accurately describe the random field, the size of the stochastic element is taken to be less than one-half of the correlation length θ. A sample tensile specimen with initial damage is generated following a cutting procedure. Firstly, a sample of 200×200 mm^2 with random initial damage is digitally generated according to the method of Yamazaki and Shinozuka (1988) [Fig. 4(a)].

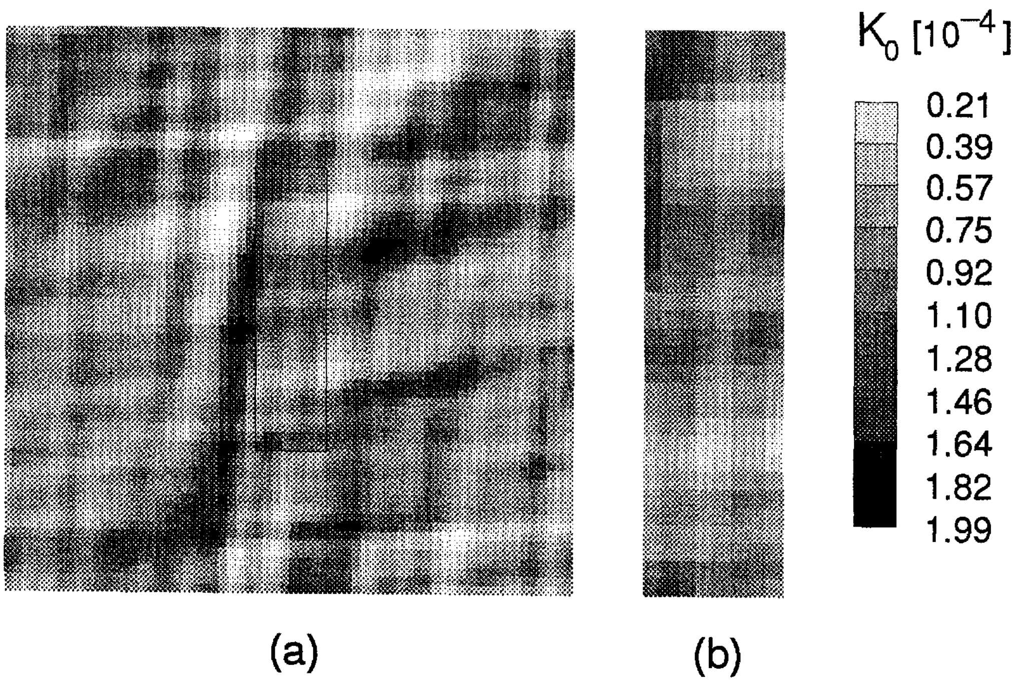


Fig. 4. (a) Typical initial damage realization for 200×200 mm^2 sample. (b) Cut-out tensile specimen (100×25 mm^2) at random position.

Out of this sample a tensile specimen with dimensions of 100×25 mm^2 is cut at a random position [Fig. 4(b)]. This cutting procedure obviates the problem of defining statistical boundary conditions or considering boundary layer effects. In the reference simulations the correlation parameter has been assigned the value $d = 5$ mm or, equivalently, a correlation length $\theta = 7$ mm.

For the finite element discretization two different meshes have been used: 8×32 and 16×64 elements. During the discretization process, the size of the stochastic element is not changed. This means that a stochastic element is either a block of one or of four finite elements with identical random properties. Furthermore, we assume a constant linear softening diagram with a softening modulus $h = -0.1E$ and an elastic modulus $E = 20{,}000$ MPa. The assumption of a deterministic softening modulus has been relaxed in Carmeliet and Hens (1994), where the initial damage threshold and softening modulus are considered as correlated random variables.

Figure 5(a) shows the deformation pattern at maximum load for the standard damage model. We observe a spurious localization pattern with the localization concentrated in a single band of elements which generally follows the mesh lines and occasionally jumps from one row to the next. The finite element solution tries to capture a line crack and the results are clearly mesh-dependent. This observation is in line with earlier findings using deterministic standard, rate-independent models (Sluys, 1992). Moreover, we observe that upon mesh refinement completely different deformation patterns can be obtained [Fig. 5(b)]. These examples indicate that the introduction of the correlation length θ does not resolve the improper behavior of the standard continuum and that regularization techniques like the nonlocal damage concept have to be used.

For the calculations with the nonlocal damage model, we choose the internal length scale $l = 5$ mm. Figure 6(a) and (b) shows the deformation patterns at maximum load and at complete failure, respectively. Before the maximum load is reached, the damage process is characterized by diffuse microcracking and by the development of two noncritical damage zones. Comparing Figs 4(b) and 6(a), we observe that these damage zones arise at the areas of the lowest initial damage threshold, which are periodically located over the specimen due to the correlation coefficient function. Note that in Fig. 4(b) a light shade means a low value of the initial damage threshold, or equivalently, a low value of the tensile strength. Damage initiates and localizes in these zones and results in a high level of damage in Fig. 6(a) (dark shades).

In contrast to the local damage model, the damage now does not proceed along the element lines and is no longer confined to one row of elements. The damage is spread out over a damage zone independent of the degree of mesh refinement. The width of the damage zone w is almost 17 mm. This is much lower than calculated by the deterministic nonlocal damage model (see Fig. 3), which can be explained by the fact that for the stochastic nonlocal damage model the width of the damage zone is related both to the internal length scale l and the correlation length θ, while for the deterministic nonlocal damage model the width of the damage zone is exclusively determined by the internal length scale l. The essence of a nonlocal definition of damage is the introduction of a spreading-out mechanism of damage, avoiding a strain localization in one line or plane of finite elements. In the random continuum, the spreading-out will be strongly influenced by the periodically located areas with a high initial damage threshold. The areas with a high initial damage threshold adjacent to weaker areas, where the damage will initiate, act as barriers for the spreading-out of damage. The effectiveness of this "barrier mechanism" depends on the value of the internal length scale l. If an internal length scale is chosen which is much larger than the correlation length θ, the correlative random structure may gradually vanish due to the averaging effects of the nonlocal model. In this case the spreading-out mechanism will prevail and a damage zone of a size equivalent to the deterministic model will be found.

Figure 7, which gives the overall stress–displacement response of the tensile specimen for a typical initial damage distribution, confirms that the results of the nonlocal stochastic model are insensitive with respect to the discretization, as the simulations with 256 and 1024 elements yield the same result. Comparing Figs 2 and 7, we observe a striking difference in the tail of the softening branch. While for the deterministic nonlocal damage model (Fig.

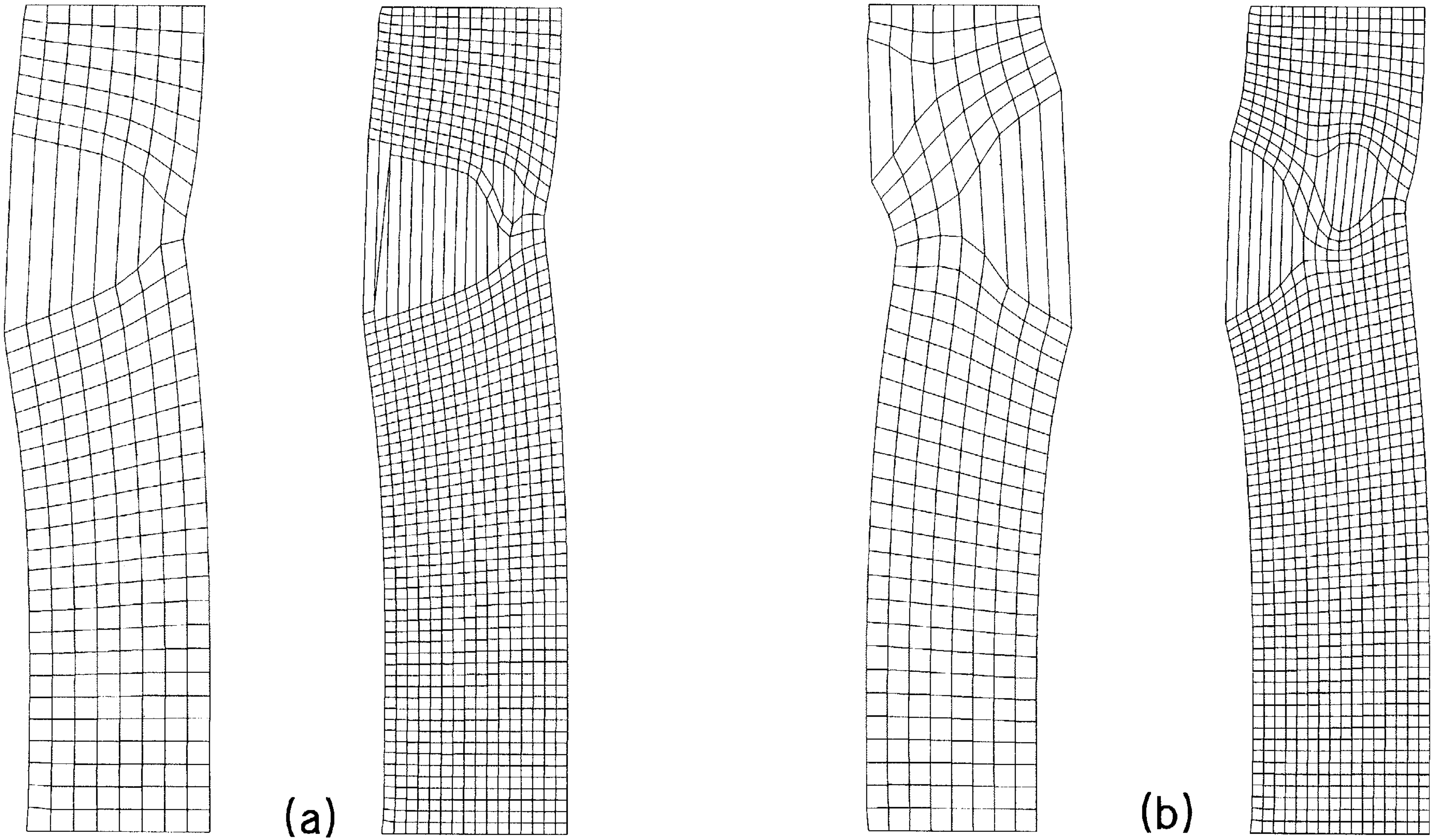


Fig. 5. Local damage model. Displacements of two tensile specimens at maximum load.

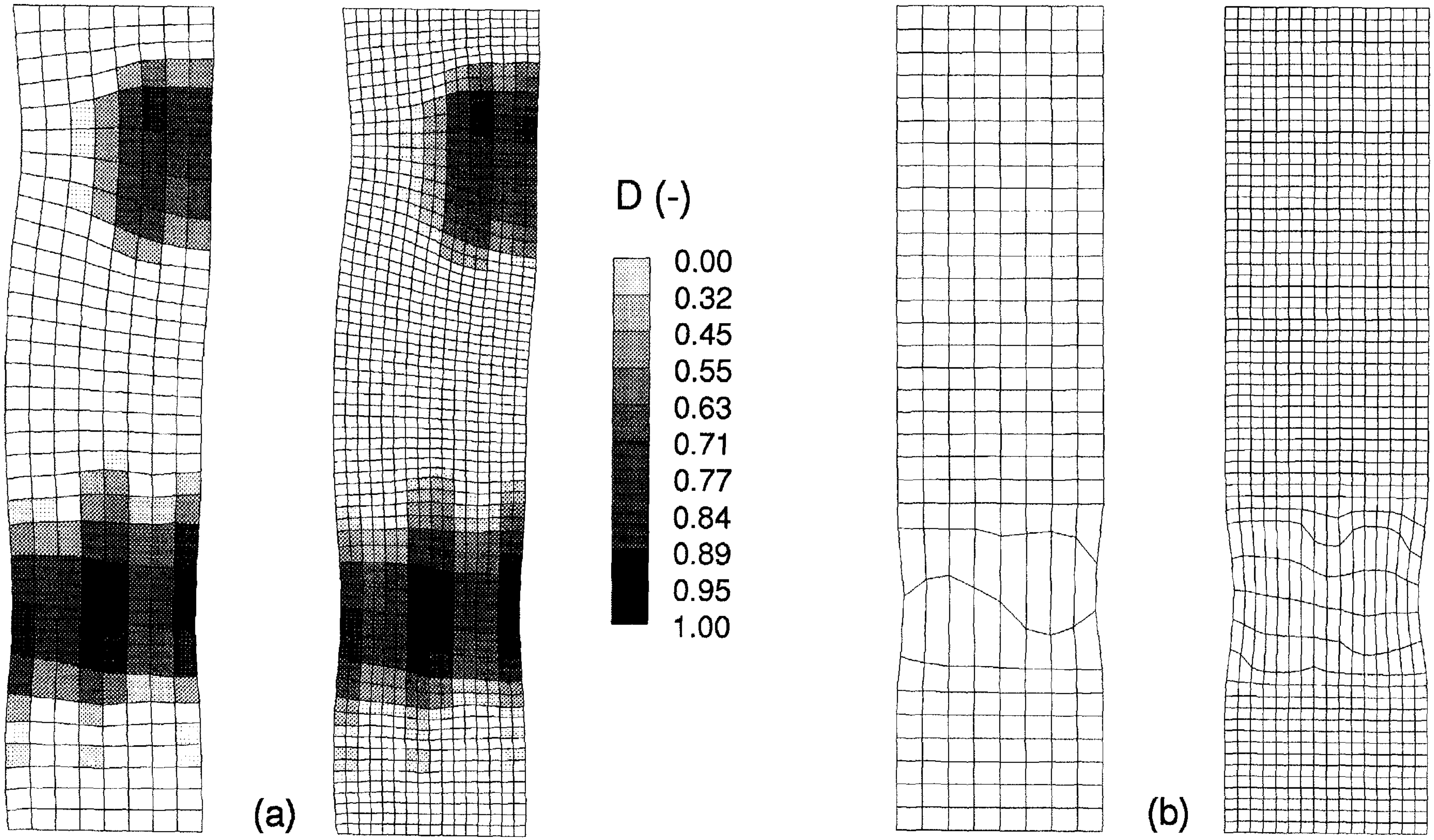


Fig. 6. Nonlocal damage model. (a) Damage distribution of tensile specimen at maximum load. (b) Displacements at complete failure.

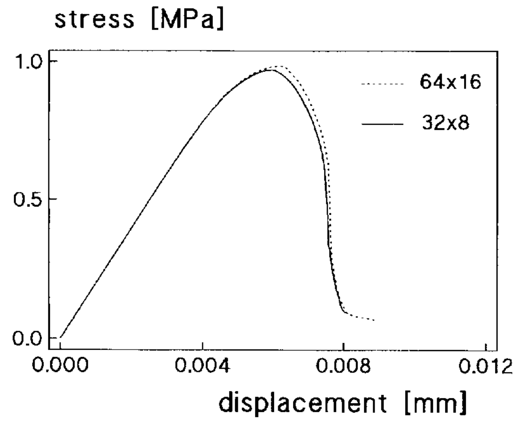


Fig. 7. Stress–displacement diagram of tensile specimen with typical initial damage distribution.

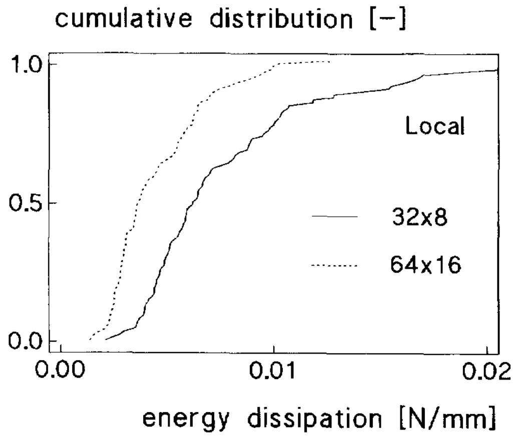
2) a steep drop is observed, the stochastic nonlocal damage model shows a much more gradual decrease of the load-carrying capacity. This shows once again that the measured phenomena in degrading materials are often structural effects (e.g. Rots and de Borst, 1989; Hordijk, 1991). It may be that the experimentally observed long tail in the softening branch is caused by the stochastic nature of heterogeneous materials.

The difference between the local and nonlocal stochastic models becomes also clear when comparing the total energy dissipation during damage, defined as $\int \sigma \, du$ with u the inelastic displacement, for the two different finite element discretizations. Figure 8 shows the cumulative distributions calculated from the responses of 100 samples using the Monte-Carlo technique. The results for the local damage model, obtained for a less steep softening branch, $h = -0.01E$, are clearly mesh dependent with decreasing energy dissipation upon mesh refinement. On the contrary, the results for the nonlocal stochastic model show a perfect agreement for both discretizations. These observations correspond fully with the findings for deterministic models.

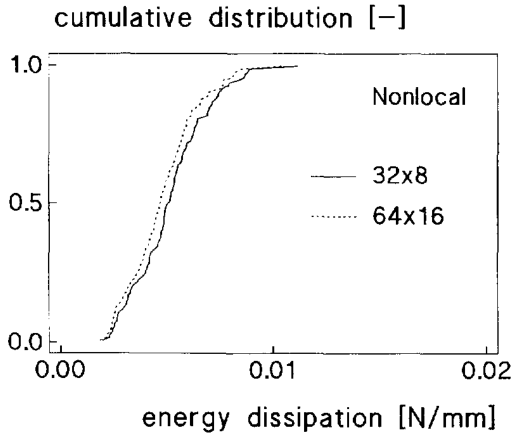
So far, we have shown that a stochastic continuum description of damage localization must include a regularization technique to prevent loss of well-posedness of the rate boundary value problem. A major problem now lies in combining the two different length parameters introduced in a physically realistic manner: the internal length scale l of the nonlocal continuum and the correlation length θ of the random field. Both length parameters result from the transition of a micro-level to a macro continuum level. While the internal length l depends merely on the typical size of defects (or aggregates), the correlation length θ depends on the size as well as on the frequency, that is, the distance between successive defects. This important issue will be illustrated by means of an example. Figure 9 compares the cumulative distribution of the energy dissipation for various values of the internal length scale of the nonlocal model ($l = 5$ and 2.5 mm) and of the correlation parameter ($d = 5$ and 10 mm). Otherwise, the standard material data have been used. The figure suggests that variations of l have a larger influence on the cumulative distribution of the energy dissipation than the correlation length θ.

4. CONCLUSION

It has been shown that caution has to be exercised with respect to results of stochastic damage models, which rely on a stochastic formulation of the classical continuum framework. A stochastic description of the damage threshold does not solve the difficulties associated with strain-softening in a standard continuum, as has been demonstrated by the example of a direct tensile test. The rate boundary value problem becomes ill-posed at the onset of localization and as a consequence spurious localization patterns and a severe mesh dependence are obtained. In order to formulate a stochastic continuum model that properly describes the progressive damage in quasi-brittle materials, both a regularization of the standard continuum and a stochastic description of the disordered continuum must be



(a)



(b)

Fig. 8. Cumulative distribution of energy dissipation during damage. (a) Local damage model: $h = -0.01\ E$; (b) nonlocal damage model: $h = -0.1\ E$.

used. In both formulations a length parameter is introduced: the internal length scale of the nonlocal continuum and the correlation length of the random field. Both length parameters rely on the same physical background: microstructural phenomena situated at a level below the continuum level. An important issue is the proper modeling of the relation

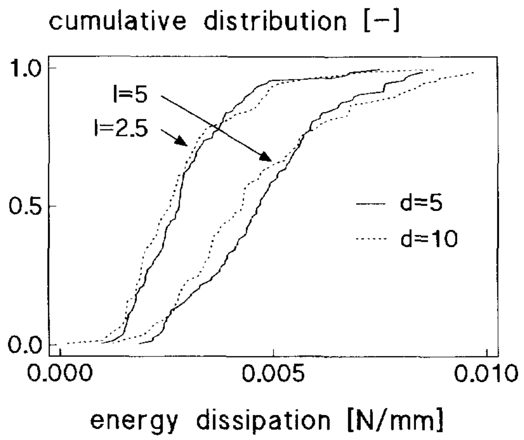


Fig. 9. Cumulative distribution of energy dissipation during damage for nonlocal continuum. Influence of the internal length scale l and correlation parameter d.

between the internal length scale and correlation length. Closely related to this is the proper experimental determination of these parameters.

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